ON A DEFORMED QUANTUM POTENTIAL

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1. BACKGROUND

In the rather incomplete survey paper [16] among other things we overlooked the fact that the Schrödinger SE) for a modified Riemann-Liouvill (MRL) calculus in [28] actually gives rise to a "heuristic" quantum potential (QP) directly as follows. The SE for MRL from [28], 41 (2009), 1590-1604 involves

$$i\hbar\psi_t^\alpha = -\frac{\hbar^2}{2m}\rho^2(\alpha)(xt)^{2(\alpha-1)}\psi_{xx} + V\psi \tag{1.1}$$

where $\rho(\alpha) = \alpha![(1-\alpha)!]^2$ and the momentum $p_\alpha \sim m\dot{x}^{(\alpha)} = mu_\alpha$ where

$$u_\alpha = \frac{(dx)^\alpha}{dt} = \alpha!\left(\frac{d^\alpha t}{dx}\right)^{-1} \tag{1.2}$$

Thus $\dot{x}^\alpha$ is defined by the α-derivative of time with respect to space (recall $\alpha! d^\alpha t = dt$) and consequently

$$u_\alpha = \rho(\alpha)(xt)^{(\alpha-1)}x^\alpha(t) \tag{1.3}$$

(cf. [16, 28]). Then when considering p_α^2 in the Hamiltonian one simply uses u_α^2. It is not immediately clear why this is a better determination of velocity than e.g. (**1A**) $v_\alpha(t) = d^\alpha x/(dt)^\alpha = \alpha![dx/(dt)^\alpha]$. Also, given a classical time variation one could also imagine a SE $(D^{2\alpha} = D^\alpha D^\alpha)$ $(D^{2\alpha} = D^\alpha D^\alpha)$

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}D^{2\alpha}\psi + V\psi \tag{1.4}$$

arising via ($\bullet$) $\psi = Rexp[iS/\hbar]$ or $\psi = RE_\alpha(z^\alpha)$ (where E_α is the Mittag-Leffler function (**1B**) $E_\alpha = E_\alpha(z^\alpha) = \sum[z^{(\alpha k)}/(\alpha k)!$ with $D^\alpha E_\alpha = E_\alpha$). In the latter situation one would have a possible quantum potential (QP) for (1.4) of the form

$$\mathfrak{Q}_\alpha = -\frac{\hbar^2}{2m}\frac{D^{2\alpha}R}{R} \tag{1.5}$$

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and in fact it might be suggested that (1.1), with a classical time derivative, might have a QP of the form

$$\hat{\mathfrak{Q}} = -\frac{\hbar^2}{2m}\rho^2(\alpha)(xt)^{2(\alpha-1)}\frac{R_{xx}}{R} \tag{1.6}$$

(cf. [16]). Given the connection between q-deformed calculi and fractional calculi (see e.g. [9, 16, 27]), and the general interest in q-deformed physics, we want to explore the notion of a q-deformed QP in the q-deformed and fractional calculations. However we recall that the connection between a QP, osmotic velocity, and information theory provides the QP with its important intrinsic meaning and it is not clear whether this arises in the fractional or deformed situations. In particular it may not be realistic to introduce the form $iS/\hbar$ via ($\bullet$) in the fractional theory. Of course if the resulting equations have solutions they may mean something or they may suggest a more profitable direction.

REMARK 1.1. The presence of the time variable in in (1.6) shows that time makes its presence felt more strongly in the fractional or deformed context (via u_α for example). Other work in [27, 33, 43] suggests memory effects in fractal situations, etc. In particular q-deformed and fractional contests are related (cf. [27]) and given the uibiquity now of q-entropy, q-Fisher information, q-statistics, etc. it may be that the QP plays a structural role in the fractal world. ∎

2. Q-DEFORMED AND FRACTIONAL CALCULUS

We extract here first from [27] (1007.1084) and refer also to other citations in [27] for more information (especially [3]). It can be shown that the concept of q-deformed Lie algebras and the methods developed in fractional calculus are closely related and may be combined leading to a new class of fractional q-deformed Lie algebras (cf. also [9, 29, 31] for q-calculus). In order now to describe a deformed Lie algebra we introduce a parameter q and define a mapping

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}};\ lim_{q\to 1}[x]_q = x;\ [0]_q = 0 \tag{2.1}$$

$$D_x^q f(x) = \frac{f(qx) - f(q^{-1}x)}{(q-q^{-1})x};\ D_x^q x^n = [n]_q x^{n-1} \tag{2.2}$$

Alternate constructions are indicated below (cf. [3, 9, 31]). As an example of a q-deformed Lie algebra one looks at the harmonic oscillator in [27]. The creation and annihilation operators $a^\dagger$, a and the number operator n generate the algebra (**2A**) $[N, a^\dagger] = a^\dagger;\ [N, a] = -a;\ aa^\dagger - q^{\pm 1}a^\dagger a = q^{\mp N}$. Via (2.1) an alternative definition for (**2A**) is given via ($\bullet\bullet\bullet$) $a^\dagger a = [N]_q$

with $aa^\dagger = [N+1]_q$. One defines a vacuum state with $a|0>=0$ and the action of the operators $\{a, a^\dagger, N\}$ on the basis $|n>$ of a Fock space is given by

$$(2.3)\quad N|n>=n|n>;\ a^\dagger|n>=\sqrt{[n+1]_q}\,|n+1>;\ a|n>=\sqrt{[n]_q}|n-1>$$

The Hamiltonian of the q-deformed harmonic oscillator and its eigenvectors are defined via

$$H=\frac{\hbar\omega}{2}(aa^\dagger+a^\dagger a);\ E^q(n)=\frac{\hbar\omega}{2}([n]_q+[n+1]_q) \tag{2.4}$$

In [27] various fractional derivatives are recalled and in particular one mentions the Caputo derivative

$$D_x^\alpha=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\int_0^x dx(x-\xi)^{-\alpha}\partial_\xi f(\xi) & 0\leq\alpha<1\\ \frac{1}{\Gamma(2-\alpha)}\int_0^x d\xi(x-\xi)^{1-\alpha}(\partial^2 f(\xi)/\partial\xi^2) & 1\leq\alpha<2\end{cases} \tag{2.5}$$

Then for $x^{n\alpha}$

$$D_x^\alpha x^{n\alpha}=\begin{cases}\frac{\Gamma(1+n\alpha)x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)} & n>0\\ 0 & n=0\end{cases} \tag{2.6}$$

The fractional derivative parameter α can be interpreted as a deformation parameter via $|n>=x^{n\alpha}$ and

$$[n]_\alpha|n>=\begin{cases}\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}|n> & n>0\\ 0 & n=0\end{cases};\ lim_{\alpha\to 1}[n]_\alpha=n \tag{2.7}$$

Then via (**2A**)) the standard q-numbers can be defined more or less heuristically and there are different possibilities. On the other hand the q-deformation based on a fractional calculus α is uniquely determined once a set of basis vectors is given and the harmonic oscillator can be used as an illustration. This means that information about q-entropy or q-information can be connected to a $\alpha-$fractional background.

In order to present a more "standard" picture we go to [27] for a free particle SE (cf. also [33]). Thus replace x and p by $\hat{x}$ and $\hat{p}$ to get into QM and write (**2B**) $\hat{X}f(x)=xf(x);\ \hat{P}f(x)=-i\hbar\partial_x f(x);\ [\hat{X},\hat{P}]=i\hbar$. Then using D_x^α there follows

$$\hat{x}=\left(\frac{\hbar}{mc}\right)^{1-\alpha}x^\alpha;\ \hat{p}=-i\left(\frac{\hbar}{mc}\right)^\alpha mcD_x^\alpha \tag{2.8}$$

The classical and quantum Hamiltonians are now

$$H_{class}=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2;\ H^\alpha=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2 \tag{2.9}$$

and the Schrödinger Hamiltonian becomes
(2.10)

$$H^\alpha\psi = \left[-\frac{1}{2m}\left(\frac{\hbar}{mc}\right)^2 m^2c^2D_x^\alpha D_x^\alpha + \frac{1}{2}m\omega^2\left(\frac{\hbar}{mc}\right)^{2(1-\alpha)}x^{2\alpha}\right]\psi = E\psi$$

which is of the type (1.4) (with $D^{2\alpha} = D^\alpha D^\alpha$). The "Hermiticity" of such an operator will depend of course on the choice of fractional derivative and it can be shown that the Feller and Riesz fractional derivatives (but not Caputo or Riemann-Liouville) will insure Hermiticity (cf. [33, 27]). Here the Riesz derivative is
(2.11)

$$D_R^\alpha f(x) = \Gamma(1+\alpha)\frac{Sin(\pi\alpha/2)}{\pi}\int_0^\infty \frac{f(x+\xi)-2f(x)+f(x-\xi)}{\xi^{\alpha+1}}d\xi;\ (0<\alpha<2)$$

and the Feller derivative is
(2.12)

$${}_FD_1^\alpha f(x) = \Gamma(1+\alpha)\frac{Cos(\pi\alpha/2)}{\pi}\int_0^\infty \frac{f(x+\xi)-f(x-\xi)}{\xi^{\alpha+1}}d\xi;\ (0\leq\alpha<1)$$

For a canonical picture a scaled energy E^q and coordinates can be introduced via

$$\xi^\alpha = \sqrt{\frac{m\omega}{\hbar}}\left(\frac{\hbar}{mc}\right)^{1-\alpha}x^\alpha;\ E = \hbar\omega E^\alpha \tag{2.13}$$

leading to the eigenvalues for H^α

$$H^\alpha\psi_n(\xi) = \frac{1}{2}\left[-D_\xi^{2\alpha}+\xi^{2\alpha}\right]\psi_n(\xi) = E'(n,\alpha)\psi_n(\xi) \tag{2.14}$$

Laskin (cf. [33]) has derived an approximate analytic solution within the framework of the WKB approximation which has the advantage of being independent of the choice of a specific definition of the fractional derivatives (cf. [27, 33] for more information on this) and the result is

$$E'(n,\alpha) = \left[\frac{1}{2}+n\right]^\alpha \pi^{\alpha/2}\left[\frac{\alpha\Gamma\left(\frac{1+\alpha}{2\alpha}\right)}{\Gamma(1/2\alpha)}\right];\ n = 0,1,2,\cdots \tag{2.15}$$

(cf. also [25, 33, 43, 44]). We write also for a harmonic oscillator (following [27]) the connection to q-deformation arises from $n >\sim x^{n\alpha}$ with (♦♦) $E^q(n) = (\hbar\omega/2)([n]_q + [n+1]_q)$ while

$$[n]_\alpha|n> = \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}|n>;\ (\alpha\sim q) \tag{2.16}$$

for $n > 0$ (as in (2.7)-(2.8)). The q-deformation is then uniquely defined by α via action on pure states.

REMARK 2.1. There is already however a huge literature on Fisher

thermodynamics (TD), q-entropy, q-Fisher information, non-extensive q statistics, multi-scale problems, etc., some of which we looked at or referred to in [16]. More synthesis and organization is needed. ■

3. Q DEFORMATION

We have discussed fractals and fractional calculi in [16] and refer also to [4, 5, 6, 7]. Some relations between fractional calculus and q-deformed calculus were used but we want to expand upon the q-deformation framework here since the relations to quantum groups etc. seem more "intrinsic" in dealing with Schrödinger equations (SE) and the quantum potential (QP) than simple recourse to fractional derivatives (cf. Remark 3.1). We will rely on [3, 9, 31] for background formulas and extract first a few formulas from [3]. Thus one defines q-numbers (**3A**) $[x] = [(q^x - q^{-x})/(q - q^{-1})]$ so $[1] = 1,\ [2] = q + q^{-1}\ [3] = q^2 + 1 + q^{-2}$, etc. and notes that (**3B**) $e_q(az) = \sum_0^\infty (a^n/[n]!)z^n$. Then

$$D_x^q f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z} \tag{3.1}$$

$$D_x^q (az^n) = a[n]z^{n-1};\ D_z^q e_q(az) = ae_q(az) \tag{3.2}$$

Further

$$\begin{aligned} D_x^q[f(x)g(x)] &= [D_x^q f(x)]g(q^{-1}x) + f(qx)[D_x^q g(x)] \\ &= (D_x^q g(x)f(q^{-1}x) + g(qx)[D_x^q f(x)] \end{aligned} \tag{3.3}$$

$$D_x^q f(x^n) = [n]x^{n-1} D_{x^n}^{q^n}(f(x^n); \tag{3.4}$$

$$D_x^{q^n} f(x) = \frac{1}{[n]} \sum_0^{n-1} D_x^q f(q^{2k-(n-1)}x)$$

$$\int_0^a f(x)d_q x = a(q^{-1} - q) \sum_0^\infty q^{2n+1} f(q^{2n+1}a) \tag{3.5}$$

$$D_x^q \int f(x)d_q x = f(x) = \int D_x^q f(x) d_q x \tag{3.6}$$

There are also Q numbers (**3C**) $[x]_Q = [(Q^x - 1)/(Q - 1)]$ with (**3D**) $[x] = q^{1-x}[x]_Q$ when $Q = q^2$. The exponential function is defined with a formula (**3E**) $exp_Q(ax) = \sum_0^\infty [a^n x^n/[n]_Q!]$ with $e_Q(x)e_{1/Q}(-x) = 1$. The Q derivative is

$$D_x^Q f(x) = \frac{f(Qx) - f(x)}{(Q - 1)x} \tag{3.7}$$

and

$$D_x^Q x^n = \frac{Q^n x^n - x^n}{(Q-1)x} = [n]_Q x^{n-1};\ [n]_Q = \frac{Q^n-1}{Q-1} \tag{3.8}$$

while (**3F**) $D_x^Q e_Q(ax) = ae_Q(ax)$. There are Leibnitz rules

$$D_x^Q[f_1(x)f_2(x)] = [D_x^Q f_1(x)]f_2(Qx) + f_1(x)[D_x^Q f_2(x)]; \tag{3.9}$$
$$D_x^Q[(f_1(x)f_2(x)] = [(D_x^Q)f_1(x)]f_2(x) + f_1(Qx)[D_x^Q f_2(x)]$$

and for a quotient

$$D_x^Q \frac{f_1(x)}{f_2(x)} = \frac{[D_x^Q f_1(x)]f_2(x) - f_1(x)[D_x^Q f_2(x)]}{f_2(Qx)f_2(x)} \tag{3.10}$$

For the second derivative one has also

$$[(D_x^Q)^2 f(x)] = \tag{3.11}$$
$$= (Q-1)^{-2}Q^{-1}x^{-2}[f(Q^2x) - (Q+1)f(Qx) + Qf(x)]$$

This leads to

$$(D_x^Q)^n f(x) = \tag{3.12}$$
$$= (Q-1)^{-n}Q^{-n(n-1)2}x^{-n}\sum_0^n \left[\begin{array}{c} n \\ k \end{array}\right]_Q (-1)^k Q^{k(k-1)/2} f(Q^{n-k}x)$$

Further

$$\int_0^1 f(x)d_Q x = (1-Q)\sum_0^\infty f(Q^k)Q^k \tag{3.13}$$

It is also easily checked that (**3G**) $D^Q \int f(x)d_Q x = f(x)$. We add to this a formula for $(D_x^q)^2$; thus

(3.14)

$$D_x^q(D_x^q f(x)) = D_x^q\left[\frac{f(qx) - f(q^{-1}x)}{(q-q^{-1})x}\right] = D_x^q h(x) = \frac{h(qx) - h(q^{-1}x)}{(q-q^{-1})x}$$

Consequently

$$(D_x^q)^2 f(x) = \frac{1}{(q-q^{-1})x}\left[\frac{f(q^2x) - f(x)}{(q-q^{-1})qx} - \frac{f(x) - f(q^{-2}x)}{(q-q^{-1})q^{-1}x}\right] = \tag{3.15}$$
$$= \frac{1}{(q-q^{-1})x^2}\left[\frac{f(q^2x)}{q^2-1} + \frac{f(q^{-2}x)}{1-q^{-2}} - f(x)\left(\frac{1}{(q^2-1)} + \frac{1}{(1-q^{-2})}\right)\right]$$

to compare with (3.11) for the Q derivative. On the other hand we have from (3.4B) for $n=2$

$$D_x^{q^2} f(x) = \frac{1}{q+q^{-1}}\left[\frac{f(x) - f(q^{-2}x)}{(q-q^{-1})x} + \frac{f(q^2x) - f(x)}{(q-q^{-1})x}\right] = \tag{3.16}$$
$$= \frac{1}{(q^2-q^{-2})x}\left[f(q^2x) - f(q^{-2}x)\right]$$

It is then appropriate to write down a SE for the q-derivative as

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}D_x^q D_x^q\psi + V\psi \tag{3.17}$$

where $D^qD^q = (D^q)^2$ is specified in (3.14). Evidently $(D_x^q)^2 \neq D_x^{2q} \neq D_x^{q^2}$.

REMARK 3.1. Q calculus is a kind of time reversal of history, since $Qx \sim x + \Delta x$ with $\Delta x \sim (Q-1)x$. On the other hand q-calculus seems to have a quantum meaning (cf. [9, 31]). Thus one might expect a chain rule for Q calculus to follow from an equation

$$D_x^Q f(x) = \frac{f(Qx) - f(x)}{(Q-1)x} = \frac{f(x+\delta x) - f(x)}{\Delta x} \tag{3.18}$$

with $Qx \sim x + \Delta x$ and $\Delta x \to 0$ when $Q \to 1$. Then q-calculus simply works on both sides of $q = 1$. This suggests working with $\psi = Rexp_Q(iS/\hbar)$ for the SE provided one has a chain rule for $f(x) = exp_Q(u(x))$ with $\Delta x \sim (Q-1)x$. According to [27] one can identify q-deformed derivatives with a fractional counterpart based on the action of creation and annihilation operators on pure states via a formula (in q-calculus) ($|n> \sim x^{n\alpha}$)

$$[n]_\alpha |n> = \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}\ (n > 0) \tag{3.19}$$

where (**3H**) $a|n> = \sqrt{[n]_q}|n-1>$ and $a^\dagger|n> = \sqrt{[n+1]_q}|n+1>$ in the q-theory. This suggests that one should perhaps be able to write down a fractional counterpart. We recall the notation in Section 1 (used for MRL) where $d^\alpha f = \alpha! df$ and look at

$$D_x^Q u(x) = \frac{f(Qx) - f(x)}{(Q-1)x} = \frac{\Delta f}{\Delta x} \sim \frac{d^\alpha f}{dx^\alpha} \tag{3.20}$$

and from [28] there is a formula

$$\frac{d^\alpha u}{du^\alpha} = (1-\alpha)! u^{\alpha-1}u^{1-\alpha} \Rightarrow (u_x'(x))^\alpha = (1-\alpha)! u^{\alpha-1}u_x^{(\alpha)}(x) \tag{3.21}$$

However from [29] there seems to be no general chain role for the Q-derivative, so let us look for something else that might illuminate matters. Thus consider for example

(3.22)
$$\frac{f(Qu(Qx)) - f(u(x))}{(Q-1)u(Q-1)x} = \frac{f(Qu(Qx)) - f(u(Qx)) + f(u,(Qx)) - f(u(x))}{(Q-1)u(Q-1)x} =$$
$$= \frac{D_u^Q(f(u(Qx))}{(Q-1)x} - \frac{D_x^Q(f(u(x))}{(Q-1)u}$$

We will pick this up again in Section 4. ■

4. DEFORMATIONS IN FRACTIONAL SPACES

We have outlined briefly in [16] some of the interaction of fractal spacetime with fractional calculus (cf. also [4, 5, 6, 7, 20, 21]) and the issue of fractional calculus and QM is also developed in [37, 38]) in terms of a dimensionally deformed D-calculus. There are some similarities of this with the Q-deformed calculus discussed above and we will survey some of this below. The physics background for [38] is omitted here and we simply note that it deals with theory for an "exciton" in a semi-conductor. The theory in [37]-1 involves a Bose-type oscillator in a fractional dimensional space and the mathematical aspects are summarized and polished in [37]-2, which we examine here. Thus we sketch a new deformed calculus (D-deformed) beginning with a 1-dimensional momentum operator (**4A**) $P = (1/i)(d/d\xi)$ where $\hbar$ is taken as 1. In a fractional space, sue to the inclusion of the integration weight

$$\frac{\sigma(D)}{2}|\xi|^{D-1};\ \sigma(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{4.1}$$

the momentum operator is no longer Hermitian. Working from the Wigner commutation relations for the canonical variables of a Bose-like oscillator one finds a momentum operator

$$P = \frac{1}{i}\frac{d}{d\xi} + i\frac{(D-1)}{2\xi}R - i\frac{D-1}{2\xi} \tag{4.2}$$

where R is a reflection operator. This suggests a deformation of QM in a fractional space via a new D-deformed derivative operator

$$\frac{d_D}{d_D\xi} = \frac{d}{d\xi} + \frac{D-1}{2\xi}(1-R) \tag{4.3}$$

with (**4B**) $P = (1/i)(d_D/d_D\xi)$. Thus the deformed annihilation and creation operators can be defined via

$$a_D = \frac{1}{\sqrt{2}}\left(\xi + \frac{d_D}{d_D\xi}\right);\ a_D^\dagger = \frac{1}{\sqrt{2}}\left(\xi - \frac{d_D}{d_D\xi}\right) \tag{4.4}$$

Then (cf. [27, 37])

$$a_D|2n> = \sqrt{2n}|2n-1>;\ a_D|2n+1> = \sqrt{2n+D}|2n>; \tag{4.5}$$

$$a_D^\dagger|2n> = \sqrt{2n+D}|2n+1>;\ a_D^\dagger|2n+1> = \sqrt{2n+2}|2n+2>$$

Now, analogous to the q or Q factor, one introduces a D factor (**4C**) $[n]_D = n + [(D-1)/2][1-(-1)^n]$ so that (4.5) can be rewritten as (**4D**) $a_D|n> = \sqrt{[n]_D}|n-1>$ and (**4E**) $a_D^\dagger|n> = \sqrt{[n+1]_D}|n+1>$. Consequently one

can define a D-deformed factorial as
(4.6)

$$[n]_D = [n]_D[n-1]_D \cdots [1]_D[0]_D! = \begin{cases} \frac{2^n(n/2)\Gamma[(n+D)/2]}{\Gamma(D/2)} & n \ even \\ \frac{2^n[(n-1)/2]\Gamma[(n+d+1)/2]}{\Gamma(D/2)} & n \ odd \end{cases}$$

The eigenstates $|n>$ can be described via

$$N_D|n>= n|n>;\ N_D = \frac{1}{2}\{a_D^\dagger, a_D\} - \frac{D}{2};\ |n>= \frac{(a_D^\dagger)^n}{\sqrt{[n]_D!}}0|> \tag{4.7}$$

One can show directly that (•) $a_D^\dagger a_D = [n]_D;\ a_D a_D^\dagger = [n+1]_D$ (cf. [27] for comparison with q and Q deformed calculi). From the definitions one can introduce now a D-deformed integration with

$$f(\xi) = \int F(\xi) d_D\xi + C;\ \frac{d_D f(\xi)}{d_D\xi} = F(\xi) \tag{4.8}$$

One expands upon this via

$$\frac{d_D f(\xi)}{d_D(\xi)} = \left[1 + \frac{D-1}{2\xi}(1-R)\int d\xi\right]\frac{df(\xi)}{d\xi} = F(\xi) \Rightarrow \tag{4.9}$$

$$\Rightarrow \frac{df(\xi)}{d\xi} = \left[1 + \frac{D-1}{2\xi}(1-R)\int d\xi\right]^{-1} F(\xi)$$

Consequently it follows that

$$f(\xi) = F(\xi)d_D\xi = \sum_0^\infty \left[-\int dx \frac{D-1}{2\xi}(1-R)\right]^n \int d\xi F(\xi) \equiv \tag{4.10}$$

$$\equiv \int F(\xi) d_D\xi = \sum_0^\infty (-1)^n I_n;\ I_{n+1} = \int \frac{D-1}{2\xi}(1-R) I_n d\xi;\ I_0 = \int F(\xi)d\xi$$

Then the following identities can be easily demonstrated

$$\frac{d_D(f(\xi)g(\xi)}{d_D\xi} = g(\xi)\frac{d_D f(\xi)}{d_D\xi} + \frac{d_D g(\xi)}{d_d\xi}Rf(\xi) + \frac{dg}{d\xi}(1-R)f(\xi) \tag{4.11}$$

(4.12)

$$\int g(\xi)\frac{d_D f(\xi)}{d_D\xi}d_D\xi = f(\xi)g(\xi) = \frac{d_D g(\xi)}{d_D\xi}Rf(\xi)d_D\xi - \int \frac{dg(\xi)}{d\xi}(1-R)f(\xi)d_D(\xi)$$

The eigenstates can now be exhibited via

$$P\Psi_p = -i\frac{d_D\psi_p}{d_D\xi} = p\Psi_p;\ \Psi_p = A_p E_D(ip\xi) \tag{4.13}$$

where $E_D(x)$ is the D-deformed exponential function. Thus

$$\frac{d_D\xi^n}{d_D\xi} = [n]_D\xi^{n-1};\ \int \xi^n d_D\xi = \frac{\xi^{n+1}}{[n+1]_D} + const. \tag{4.14}$$

(4.15)
$$E_D(\xi)=\sum_0^\infty \frac{\xi^n}{[n]_D!};\ \frac{d_D E_D(\lambda\xi)}{d_D\xi}=\lambda E_D(\lambda\xi);\ \int E_D(\lambda\xi)=\frac{E_D(\lambda\xi)}{\lambda}+c$$

For eigenstates one has (**4F**) $a_D|\alpha>=\alpha|\alpha>$ for coherent states where
(4.16)
$$|\alpha>=A_\alpha\sum_0^\infty\frac{\alpha^n}{\sqrt{[n]_D!}}|n>;\ A_\alpha=\frac{1}{\sqrt{E_D(|\alpha|^2)}};\ |\alpha>=\frac{E_D(\alpha a^\dagger]_D)|0>}{\sqrt{E_D(|\alpha|^2)}}$$

and we note that (4.7) is analogous to a formula in Q-deformation from [3], p. 547, with a Q deformed harmonic oscillator (with a background q, q^{-1} oscillator satisfying

$$a=q^{1/2}bq^{-N/2};\ a^\dagger=q^{1/2}q^{-N/2}b^\dagger \tag{4.17}$$

$$[N,a^\dagger]=a^\dagger;\ [N,a]=-a;\ aa^\dagger=-q^{\mp}a^\dagger a=q^{\pm N} \tag{4.18}$$

$$|n>=\frac{(a^\dagger)^n}{\sqrt{[n]!}}|0>;\ N|n>=n|n> \tag{4.19}$$

For the associated Q situation one has (cf. also [1, 32])

$$[N,b^\dagger]=b^\dagger;\ [N,b]=-b;\ b^\dagger b=[N]_D; \tag{4.20}$$

$$bb^\dagger=[N+1]_Q;\ b|0>=0;\ |n>=\frac{(b^\dagger)^n}{\sqrt{[n]_Q!}}|0>$$

which corresponds to a formula (4.7) with $D\sim Q$.

REMARK 4.1. This seems to be a significant connection since it shows directly that a Q deformation can be related to a dimensional factor D. The connections between q-deformed situations and fractional derivatives arising in [27] for example are somewhat more obscure with a complicated formula relating q and the fractional term α. It is perhaps suggested via $D\sim Q$ that one may begin with a fractional situation, with a deformation parameter D (as in (4.1)-(4.2)), and proceed to a quantum type oscillator with $Q\sim D$ (as in [1, 3, 32]). If so this would seem to be an important factor in modern work on QM and gravity as in [4, 5, 6, 7]. ■

5. DIFFERENTIAL EQUATIONS

We want now to develop a SE type equation in the D and Q situations. For a single degree of freedom in a fractional space, following [37]-1, one introduces a Cartesian type pseudo-coordinate ξ ($-\infty<\xi<\infty$). The radial integration weight may be written as (cf. [41])

$$\sigma(D)r^{D-1}=\frac{\sigma(D)}{2}|\xi|^{D-1};\ \sigma(D)=\frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{5.1}$$

In this way the volume of the radius R_0 sphere in the fractional space is

$$(5.2)\quad V(R,D) = \int_0^{R_0} gs(D) r^{D-1} dr = \int_{R_0}^{R_0} \frac{\sigma(D)}{2} |\xi|^{D_1} d\xi = \frac{\pi^{D/2} R_0)^D}{\Gamma(1+(D/2))}$$

One takes $\hbar = 1$ and the 1-dimensional momentum operator is then (**5A**) $P = (1/i)(d/d\xi)$ which is not Hermitian for $D \neq 1$ and one also has to reject (**5B**) $[\xi, P] = i$. The most general wave-mechanical representation of P can be found by considering the general Wigner commutation relations for a Bose-like oscillator

$$(5.3)\qquad iP = [\xi, (P^2 + \xi^2)/2];\ -i\xi = [P, (P^2 + \xi^2)/2]$$

The relations above can be rewritten as

$$(5.4)\qquad [P, S] = 0;\ [\xi, S] = 0;\ S = [\xi, P] - i$$

This leads to the following expression

$$(5.5)\qquad < \xi'|S|\xi'' > - = (\xi' - \xi) < \xi'|P|\xi'' > -i\delta(\xi' - \xi'')$$

On the other hand from (5.4)-B one has (**5C**) $< \xi'|S|\xi'' >= 2iA(\xi')\delta(\xi' + \xi'')$ where A is a complex function. This, with (5.5), shows that

$$(5.6)\qquad < \xi'|P|\xi'' >= -i\delta'(\xi' - \xi'') + i\frac{A(\xi')}{\xi'}\delta(\xi' + \xi'') + B(\xi')\delta(\xi' - \xi'')$$

Introducing the completeness condition $\int d\xi'|\xi' >< \xi'| = 1$ in (5.6) and writing the wave-function for a state $|\cdots)$ as $\psi(\xi') =< \xi', \cdots)$ it can be seen that

$$(5.7)\qquad P\psi(\xi') = -\frac{d\psi(\xi')}{d\xi'} + i\frac{A(\xi')}{\xi'}\psi(-\xi') + B(\xi')\psi(\xi')$$

Thus the most general wave-mechanical representation for the momentum operator is

$$(5.8)\qquad P = \frac{1}{i}\frac{d}{d\xi} + i\frac{A(\xi)}{\xi}R + B(\xi)$$

The S operator (**5C**) above can then be written as (**5D**) $S = 2iA(\xi)R$. From the anti-hermiticity ($S = -S^\dagger$) and via substitution of (5.8) and (**5D**) in (5.4) one arrives at restrictions on A and B of the form

$$(5.9)\qquad A^*(\xi) = A(-\xi);\ \frac{dA(\xi)}{d\xi} + i[B(\xi) + B(-\xi)]A(\xi) = 0$$

For a 1-dimensional fractional-dimensional momentum one requires the hermiticity of P (cf. [37]-3) and via (5.8) and (5.1) B must be given by (**5E**) $B(\xi) = i[(D-1)/2\xi]$. Then (**5E**) and (5.4) lead to $A(\xi) = constant$ and one can write

$$(5.10)\qquad \nabla = \frac{d}{d\xi} - \frac{A}{\xi}R + \frac{D-1}{2\xi}$$

Then requiring $\nabla u = 0 \iff u = constant$ one gets (**5F**) $A = [(D-1)/2]$ and

$$(5.11)\qquad P = \frac{1}{i}\frac{d}{d\xi} + i\frac{(D-1)}{2\xi}R - i\frac{(D-1)}{2\xi} = \frac{1}{i}\frac{d}{d\xi} + i\frac{(D-1)(R-1)}{2\xi}$$

This is essentially a Dunkle operator (cf. [18]) and R is the reflection operator which reverses sign of the argument (i.e. $R = \pm 1$, even or odd); thus
(5.12)

$$\psi^{even}(\xi) = \psi(\xi)+\psi(-\xi) = (1+R)\psi(\xi);\;\; \psi^{odd}(\xi) = \psi(\xi)-\psi(-\xi) = (1-R)\psi(\xi)$$

Note that in a 1-dimensional space the Heisenberg uncertainty becomes $\sqrt{(\Delta\xi)^2(\Delta P)^2} \geq 1/2$ and this follows from (**5B**). However in a fractional dimensional space one obtains

$$(5.13)\qquad \sqrt{((\Delta\xi)^2)}\sqrt{(\Delta p)^2)} \geq \begin{cases} D/2 & even\ states \\ (2-D)/2 & odd\ states \\ 1/2 & otherwise \end{cases}$$

(we refer to [37]-1 for details).

Now consider a free particle in a fractional-dimensional space and write the Hamiltonian as

$$(5.14)\qquad H = \frac{P^2}{2} = -\frac{1}{2}\left[\frac{d^2}{d\xi^2} + \frac{(D-1)}{\xi}\frac{d}{d\xi} - \frac{(D-1)(1-R)}{2\xi^2}\right]$$

(assuming $\hbar = m = 1$). The SE equation here is stationary in the form $H\psi = E\psi$ and the eigenfunctions satisfy

$$(5.15)\qquad \left[\frac{d^2}{d\xi^2} + \frac{(D-1)}{\xi}\frac{d}{d\xi} + 2E\right]\psi^{even}(\xi) = 0;$$

$$\left[\frac{d^2}{d\xi^2} + \frac{(D-1)}{\xi}\frac{d}{d\xi} - \frac{(D-1)}{\xi^2} + 2E\right]\psi^{odd}(\xi) = 0$$

based on $R = \pm 1$. This leads to
(5.16)

$$\psi_p(\xi) = A_p(|p\xi|^{1-(D/2)}[J_{(D/2)-1}(|p\xi|)+isgn(p\xi)J_{D/2}(|p\xi|)];\;\; A_p = \sqrt{\frac{|p|^{D-1}}{2\sigma(D)}}$$

Here $sgn(x) = 1$ $(x > 0)$ and $= -1$ $(x < 0)$ and the $J_\nu(x)$ are Bessel functions. Note from [30] (p. 32) that Bessel functions of the first kind have an expression

$$(5.17)\qquad J_\nu(z) = \sum_0^\infty \frac{(-1)^k(z/2)^{2k+\nu}}{k!\Gamma(\nu+k+1)}$$

An interesting picture of the behavior of a free particle in a fractional dimensional space may now be obtained by computing the position dependence of the probability density

$$\rho_p = \frac{\sigma(D)}{2}|\xi|^{D-1}|\psi_p|^2 \tag{5.18}$$

(cf. [37]-1 for discussion).

For the study of a fractional dimensional Bose type oscillator one puts (5.11) in $H = (P^2 + \xi^2)/2$ to obtain the Hamiltonian (cf. (5.14))

$$H = -\frac{1}{2}\left[\frac{d^2}{d\xi^2} + \frac{(D-1)}{\xi}\frac{d}{d\xi} - \frac{(D-1)-(D-1)R}{2\xi^2} - \xi^2\right] \tag{5.19}$$

Certain classes of integrable many-body systems (Calogero models) have been analyzed using the Dunkl operator leading to a Hamiltonian essentially of the form (5.19). This suggests a connection between the N-body Calogero problem and the fractional dimensional Bose-like oscillator (cf. [37]-1 for more discussion).

6. ANOTHER KIND OF QP

Given a SE based on (5.13) one might look for solutions $\psi = rexp(i\mathfrak{S}/\hbar)$ or $\psi = rE_D(i\mathfrak{S}/\hbar)$ (as in ($\bullet$) of Section 1) where E_D is the D-deformed exponential as in (4.14)-(4.15) (for $\hbar = m = 1$). We reserve R here for the reflection operator below. This also may not be realistic but we are motivated to be heuristic here. The t variable can be omitted in looking for the QP (quantum potential) and we can simply use (5.11) to express H in terms of $\mathcal{D}_D = d_D/d^D\xi = \mathcal{D}_\xi + (\alpha_D/\xi)$ with $\alpha_D = (D-1)(1-R)/2$ (cf. (4.3)). Thus, following [37]-1,

$$H = -\frac{1}{2}\left[\left(\mathcal{D}_D - \frac{\alpha_D}{\xi}\right)^2 + \frac{(D-1)}{\xi}\left(\mathcal{D}_D - \frac{\alpha_D}{\xi}\right) - \frac{\alpha_D}{\xi^2} - \xi^2\right] = \tag{6.1}$$

$$= -\frac{1}{2}\left[\mathcal{D}_D^2 - \frac{2\alpha_D}{\xi}\mathcal{D}_D + \frac{\alpha_D}{\xi^2}\mathcal{D}_D + \right.$$

$$\left. + \frac{\alpha_D^2}{\xi^2} + \frac{(D-1)}{\xi}\mathcal{D}_D - \frac{(D-1)\alpha_D}{\xi^2} - \frac{\alpha_D}{\xi^2} - \xi^2\right] =$$

$$= -\frac{1}{2}\left[\mathcal{D}_D^2 - \frac{2\alpha_D}{\xi}\mathcal{D}_D + \frac{\alpha_D}{\xi^2}\mathcal{D}_D + \frac{\alpha_D^2}{\xi^2} + \frac{(D-1)\mathcal{D}_D}{\xi} - \frac{D\alpha_D}{\xi^2} - \xi^2\right] =$$

$$= -\frac{1}{2}\left[\mathcal{D}_D^2 + \left(\frac{D-1-2\alpha_D}{\xi} + \frac{\alpha_D}{\xi^2}\right)\mathcal{D}_D + \left(\frac{\alpha_D^2 - D\alpha_D}{\xi^2}\right) - \xi^2\right]$$

We note also that (**6A**) $\alpha_D(\alpha_D - D) = (1/2)[-D(1+R) + (R-1)]$ and (**6B**) $2\alpha_D = (D-1)(1-R)$.

Now working with the ξ variable as in (5.19) we can examine a stationary SE, $H\psi = E\psi$ with $\psi = rexp(iS/\hbar)$, $S = S(\xi)$, and $\hbar = 1$ in order to generate a "putative" QP. Thus

$$\psi_\xi = r_\xi exp(\cdot) + ir_\xi S_\xi exp(\cdot) \tag{6.2}$$

$$\psi_{\xi\xi} = r_{\xi\xi} exp(\cdot) + 2ir_\xi S_\xi exp(\cdot) + irS_{\xi\xi} exp(\cdot) + r(iS_\xi)^2 exp(\cdot)$$

The real term gives then from (5.19) (cf. also (5.14))

$$(H\psi)_{real} = E\psi \Rightarrow Er \sim -\frac{1}{2}\left[r_{\xi\xi} - rS_\xi^2 + \frac{D-1}{\xi}r_\xi - \frac{\alpha_D r}{\xi^2} - \xi^2 r\right] \tag{6.3}$$

For even (resp. odd) eigenfunctions in (5.15) one has $R = 1 \sim \alpha_D = 0$ or $R = -1 \sim \alpha_D = D-1$. Thus for odd eigenfunctions (6.3) becomes (cf. (5.15))

$$H\psi_{real} = E\psi_{real} \Rightarrow Er \sim -\frac{1}{2}\left[r_{\xi\xi} - rS_\xi^2 + \frac{D-1}{\xi}r_\xi - \frac{(D-1)r}{\xi^2} - \xi^2 r\right] \tag{6.4}$$

and the kinetic energy term involves $(1/2)(S_\xi^2)$. The $\xi^2 r$ term is already a potential energy so this leaves a residual energy term

$$\mathfrak{Q} \sim -\frac{r_{\xi\xi}}{2r} - \frac{D-1}{2r\xi}r_\xi + \frac{D-1}{2\xi^2} \tag{6.5}$$

which can perhaps play the role of a quantum potential (QP). Note also for $R = -1$ and $\alpha_D = D-1$ with $\mathcal{D}_D = d_\xi + [(D-1)/\xi]$ one has

$$\mathcal{D}_D^2 = d_\xi^2 + \frac{D-1}{\xi}d_\xi - \frac{D-1}{\xi^2} + \frac{(D-1)^2}{\xi^2} = \tag{6.6}$$

$$= d_\xi^2 + \frac{D-1}{\xi}d_\xi + \frac{1}{\xi^2}(D-1)(2-D)$$

Hence

$$\frac{\mathcal{D}_D^2 r}{2r} = -\mathfrak{Q} - \frac{(D-1)(1-2D)}{2\xi^2} \tag{6.7}$$

which might be of some interest. The formula (6.7) does remind one of the "standard" formulas for a QP of the form (**6C**) $\beta(\Delta r/r)$ or $\gamma(D^{2\alpha}r/r)$. For completeness we list also the stationary state formula for the even eigenfunctions (where $\alpha_D = 0$ and $\mathcal{D}_D = d_\xi$)

$$H\psi_{real} = E\psi_{real} \sim -\frac{1}{2}\left[r_{\xi\xi} - rS_\xi^2 + \frac{D-1}{\xi}r_\xi - \xi^2 r\right] \tag{6.8}$$

Here we encounter the relation (**6D**) $\mathfrak{Q} \sim -[\mathcal{D}_D^2 r/2r]$ more in the traditional mold.

At this point it would be nice to relate the operators $\mathcal{D}_D$ and D_Q from

(3.7) and (4.3) (cf. also (4.16)-(4.20) for Q and D connections). It is interesting that the powers of ξ or x which arise in fractional and deformed calculus seem to have a natural physical role. For relations between q-deformation and generalized statistics see e.g. [16, 34, 35] and references there; see also [2, 9, 16, 31, 36, 39, 45, 46] for q or Q deformed quantum mechanics.

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